

An Approximate Solution of the Jaynes–Cummings Model with Dissipation II : Another Approach

Kazuyuki FUJII ^{*} and Tatsuo SUZUKI [†]

^{*}Department of Mathematical Sciences

Yokohama City University

Yokohama, 236–0027

Japan

[†]Department of Mathematical Sciences

College of Systems Engineering and Science

Shibaura Institute of Technology

Saitama, 337–8570

Japan

Abstract

In the preceding paper (arXiv:1103.0329 [math-ph]) we treated the Jaynes–Cummings model with dissipation and gave an approximate solution to the master equation for the density operator under the general setting by making use of the Zassenhaus expansion.

However, to obtain a compact form of the approximate solution (which is in general complicated infinite series) is very hard when an initial condition is given. To overcome

^{*}E-mail address : fujii@yokohama-cu.ac.jp

[†]E-mail address : suzukita@sic.shibaura-it.ac.jp

this difficulty we develop another approach and obtain a compact approximate solution when some initial condition is given.

This paper is a sequel to the preceding one [1]. In the paper we treat the Jaynes–Cummings model with dissipation (or the quantum damped Jaynes–Cummings model in our terminology) once more and study the structure of general solution from a mathematical point of view.

We want to apply it to Quantum Computation and Quantum Control which are our final target [2], [3]. As a general introduction to these topics see for example [4] and [5].

We expect that our study will become a starting point to study more sophisticated models with dissipation in a near future.

Let us start with the following phenomenological master equation for the density operator of the atom–cavity system in [6] :

$$\frac{\partial}{\partial t}\rho = -i[H, \rho] + \mu \left\{ a\rho a^\dagger - \frac{1}{2}(a^\dagger a\rho + \rho a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho a - \frac{1}{2}(aa^\dagger \rho + \rho aa^\dagger) \right\} \quad (1)$$

where H (for simplicity we write H not H_{JC} in [1]) is the well-known Jaynes-Cummings Hamiltonian (see [7]) given by

$$\begin{aligned} H &= \frac{\omega_0}{2}\sigma_3 \otimes \mathbf{1} + \omega_0 \mathbf{1}_2 \otimes a^\dagger a + \Omega (\sigma_+ \otimes a + \sigma_- \otimes a^\dagger) \\ &= \begin{pmatrix} \frac{\omega_0}{2} + \omega_0 N & \Omega a \\ \Omega a^\dagger & -\frac{\omega_0}{2} + \omega_0 N \end{pmatrix} \end{aligned} \quad (2)$$

with

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

, and a and a^\dagger are the annihilation and creation operators of an electro-magnetic field mode in a cavity, $N \equiv a^\dagger a$ is the number operator, and μ and ν ($\mu > \nu \geq 0$) are some constants depending on it (for example, a damping rate of the cavity mode).

Note that the density operator ρ is in $M(2; \mathbf{C}) \otimes M(\mathcal{F}) = M(2; M(\mathcal{F}))$, namely

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \in M(2; M(\mathcal{F})) \quad (3)$$

where $M(\mathcal{F})$ is the set of all operators on the Fock space \mathcal{F} defined by

$$\begin{aligned}\mathcal{F} &\equiv \text{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \dots\} \\ &= \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \mid \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}; \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle\end{aligned}$$

and $\mathbf{1}$ in (2) is the identity operator.

Now, we decompose (2) into diagonal and off-diagonal parts

$$H = H_d + H_{off} = \begin{pmatrix} \frac{\omega_0}{2} + \omega_0 N & \\ & -\frac{\omega_0}{2} + \omega_0 N \end{pmatrix} + \begin{pmatrix} & \Omega a \\ \Omega a^\dagger & \end{pmatrix} \quad (4)$$

and rewrite (1) as

$$\frac{\partial}{\partial t} \rho = -i[H_d, \rho] + \mu \left\{ a \rho a^\dagger - \frac{1}{2}(a^\dagger a \rho + \rho a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho a - \frac{1}{2}(a a^\dagger \rho + \rho a a^\dagger) \right\} - i[H_{off}, \rho]. \quad (5)$$

Namely, the main part is

$$-i[H_d, \rho] + \mu \left\{ a \rho a^\dagger - \frac{1}{2}(a^\dagger a \rho + \rho a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho a - \frac{1}{2}(a a^\dagger \rho + \rho a a^\dagger) \right\}$$

, while a kind of perturbed one is

$$-i[H_{off}, \rho].$$

This is the main difference between [1] and this paper. Although the form may be not standard it is very useful to calculate when some initial conditions are given.

Let us write down the equation (5) in a component-wise manner. Then

$$\begin{aligned}\dot{\rho}_{00} &= -i\omega_0(N\rho_{00} - \rho_{00}N) + \mu \left\{ a\rho_{00}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{00} + \rho_{00}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{00}a - \frac{1}{2}(a a^\dagger \rho_{00} + \rho_{00}a a^\dagger) \right\} \\ &\quad - i\Omega(a\rho_{10} - \rho_{01}a^\dagger), \\ \dot{\rho}_{01} &= -i\omega_0(\rho_{01} + N\rho_{01} - \rho_{01}N) + \mu \left\{ a\rho_{01}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{01} + \rho_{01}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{01}a - \frac{1}{2}(a a^\dagger \rho_{01} + \rho_{01}a a^\dagger) \right\} \\ &\quad - i\Omega(a\rho_{11} - \rho_{00}a), \\ \dot{\rho}_{10} &= -i\omega_0(-\rho_{10} + N\rho_{10} - \rho_{10}N) + \mu \left\{ a\rho_{01}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{01} + \rho_{01}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{01}a - \frac{1}{2}(a a^\dagger \rho_{01} + \rho_{01}a a^\dagger) \right\} \\ &\quad - i\Omega(a^\dagger \rho_{00} - \rho_{11}a^\dagger), \\ \dot{\rho}_{11} &= -i\omega_0(N\rho_{11} - \rho_{11}N) + \mu \left\{ a\rho_{11}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{11} + \rho_{11}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{11}a - \frac{1}{2}(a a^\dagger \rho_{11} + \rho_{11}a a^\dagger) \right\} \\ &\quad - i\Omega(a^\dagger \rho_{01} - \rho_{10}a)\end{aligned} \quad (6)$$

where $\dot{\rho}_{ij} = (\partial/\partial t)\rho_{ij}$ as usual.

Here we use some technique used in [1] (see also [8] and [9]), which is very useful in some case. For a matrix $X = (x_{ij}) \in M(\mathcal{F})$

$$X = \begin{pmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we correspond to the vector $\hat{X} \in \mathcal{F}^{\dim_{\mathcal{C}} \mathcal{F}}$ as

$$X = (x_{ij}) \longrightarrow \hat{X} = (x_{00}, x_{01}, x_{02}, \cdots; x_{10}, x_{11}, x_{12}, \cdots; x_{20}, x_{21}, x_{22}, \cdots; \cdots)^T \quad (7)$$

where T means the transpose. Then the following formula

$$\widehat{EXF} = (E \otimes F^T) \hat{X} \quad (8)$$

holds for $E, F, X \in M(\mathcal{F})$.

This and equations (6) give

$$\begin{aligned} \dot{\hat{\rho}}_{00} &= \left[-i\omega_0 (N \otimes \mathbf{1} - \mathbf{1} \otimes N) + \mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2}(a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \right. \\ &\quad \left. \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2}(aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{00} - i\Omega (a \otimes \mathbf{1} \hat{\rho}_{10} - \mathbf{1} \otimes (a^\dagger)^T \hat{\rho}_{01}), \\ \dot{\hat{\rho}}_{01} &= \left[-i\omega_0 (\mathbf{1} \otimes \mathbf{1} + N \otimes \mathbf{1} - \mathbf{1} \otimes N) + \mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2}(a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \right. \\ &\quad \left. \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2}(aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{01} - i\Omega (a \otimes \mathbf{1} \hat{\rho}_{11} - \mathbf{1} \otimes a^T \hat{\rho}_{00}), \\ \dot{\hat{\rho}}_{10} &= \left[-i\omega_0 (-\mathbf{1} \otimes \mathbf{1} + N \otimes \mathbf{1} - \mathbf{1} \otimes N) + \mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2}(a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \right. \\ &\quad \left. \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2}(aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{10} - i\Omega (a^\dagger \otimes \mathbf{1} \hat{\rho}_{00} - \mathbf{1} \otimes (a^\dagger)^T \hat{\rho}_{11}), \\ \dot{\hat{\rho}}_{11} &= \left[-i\omega_0 (N \otimes \mathbf{1} - \mathbf{1} \otimes N) + \mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2}(a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \right. \\ &\quad \left. \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2}(aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{11} - i\Omega (a^\dagger \otimes \mathbf{1} \hat{\rho}_{01} - \mathbf{1} \otimes a^T \hat{\rho}_{10}) \end{aligned} \quad (9)$$

because $\mathbf{1}$ and $N = a^\dagger a$ are diagonal ($\mathbf{1}^T = \mathbf{1}$, $N^T = N$).

Next, in order to rewrite matrix elements in terms of Lie algebraic notations used in [8] we set

$$\begin{aligned} K_+ &= a^\dagger \otimes a^T, \quad K_- = a \otimes (a^\dagger)^T, \quad K_3 = \frac{1}{2}(N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}), \\ K_0 &= N \otimes \mathbf{1} - \mathbf{1} \otimes N. \end{aligned} \quad (10)$$

Then it is easy to see

$$\begin{aligned} (K_+)^{\dagger} &= K_-, \quad (K_3)^{\dagger} = K_3, \quad (K_0)^{\dagger} = K_0, \\ [K_3, K_+] &= K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3, \\ [K_0, K_+] &= [K_0, K_-] = [K_0, K_3] = 0. \end{aligned} \quad (11)$$

Namely, $\{K_+, K_-, K_3\}$ are generators of the Lie algebra $su(1, 1)$, see for example [10] as a general introduction.

If we set from (3)

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \implies \hat{\rho} = \begin{pmatrix} \hat{\rho}_{00} \\ \hat{\rho}_{01} \\ \hat{\rho}_{10} \\ \hat{\rho}_{11} \end{pmatrix} \quad (12)$$

we obtain the following “canonical” form

$$\frac{\partial}{\partial t} \hat{\rho} = (X + Y) \hat{\rho} \quad (13)$$

with

$$\begin{aligned} X &= \begin{pmatrix} -i\omega_0 K_0 + L & 0 & 0 & 0 \\ 0 & -i\omega_0 - i\omega_0 K_0 + L & 0 & 0 \\ 0 & 0 & i\omega_0 - i\omega_0 K_0 + L & 0 \\ 0 & 0 & 0 & -i\omega_0 K_0 + L \end{pmatrix}, \\ Y &= -i\Omega \begin{pmatrix} 0 & -\mathbf{1} \otimes (a^\dagger)^T & a \otimes \mathbf{1} & 0 \\ -\mathbf{1} \otimes a^T & 0 & 0 & a \otimes \mathbf{1} \\ a^\dagger \otimes \mathbf{1} & 0 & 0 & -\mathbf{1} \otimes (a^\dagger)^T \\ 0 & a^\dagger \otimes \mathbf{1} & -\mathbf{1} \otimes a^T & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} L &= \mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2}(a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2}(aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \\ &= \frac{\mu - \nu}{2} + \nu K_+ + \mu K_- - (\mu + \nu) K_3 \end{aligned}$$

where the fundamental relation $aa^\dagger = a^\dagger a + \mathbf{1} = N + \mathbf{1}$ and a simplified notation ω_0 in place of $\omega_0 \mathbf{1} \otimes \mathbf{1}$ have been used.

Let us note once more that X is not anti-hermitian, while Y is anti-hermitian.

Since the general solution of the equation (13) is given by

$$\hat{\rho}(t) = e^{t(X+Y)} \hat{\rho}(0) \quad (14)$$

in a formal way we must calculate the term $e^{t(X+Y)}$, which is in general not easy (to obtain a compact form is almost impossible). For that the following Zassenhaus formula is convenient.

Zassenhaus Formula We have an expansion

$$e^{t(A+B)} = \dots e^{-\frac{t^3}{6}\{2[[A,B],B]+[[A,B],A]\}} e^{\frac{t^2}{2}[A,B]} e^{tB} e^{tA}. \quad (15)$$

The formula is a bit different from that of [11].

In this paper we use the approximation

$$e^{t(X+Y)} \approx e^{\frac{t^2}{2}[X,Y]} e^{tY} e^{tX}. \quad (16)$$

Let us calculate each term explicitly.

[I] First, we calculate e^{tX} . The result is

$$e^{tX} = \begin{pmatrix} e^{t(-i\omega_0 K_0 + L)} & 0 & 0 & 0 \\ 0 & e^{-i\omega_0 t} e^{t(-i\omega_0 K_0 + L)} & 0 & 0 \\ 0 & 0 & e^{i\omega_0 t} e^{t(-i\omega_0 K_0 + L)} & 0 \\ 0 & 0 & 0 & e^{t(-i\omega_0 K_0 + L)} \end{pmatrix} \quad (17)$$

and fortunately in [8] the term $e^{t(-i\omega_0 K_0 + L)}$ has been calculated exactly. Namely,

$$e^{t(-i\omega_0 K_0 + L)} = e^{\frac{\mu - \nu}{2} t} e^{G(t) K_+} e^{-i\omega_0 t K_0 - 2 \log(F(t)) K_3} e^{E(t) K_-} \quad (18)$$

where

$$\begin{aligned}
E(t) &= \frac{\frac{2\mu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)}{\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)}, \\
F(t) &= \cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right), \\
G(t) &= \frac{\frac{2\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)}{\cosh\left(\frac{\mu-\nu}{2}t\right) + \frac{\mu+\nu}{\mu-\nu} \sinh\left(\frac{\mu-\nu}{2}t\right)}.
\end{aligned} \tag{19}$$

This is a kind of disentangling formula, see for example [10] as a general introduction.

If from (18)

$$\widehat{\tau}(t) \equiv e^{t(-i\omega_0 K_0 + L)} \widehat{\tau}(0) = e^{\frac{\mu-\nu}{2}t} e^{G(t)K_+} e^{-i\omega_0 t K_0 - 2\log(F(t))K_3} e^{E(t)K_-} \widehat{\tau}(0) \tag{20}$$

then the original form is given by

$$\begin{aligned}
\tau(t) &= \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \{ \exp(\{-i\omega_0 t - \log(F(t))\}N) \times \\
&\quad \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \tau(0) (a^\dagger)^m \right\} \exp(\{i\omega_0 t - \log(F(t))\}N) \} a^n.
\end{aligned} \tag{21}$$

See [8]. Next, we list some results from [9] for the latter convenience.

(i) If $\tau(0) = |0\rangle\langle 0|$ then

$$\tau(t) = \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} e^{\log G(t)N}. \tag{22}$$

(ii) If $\tau(0) = |\alpha\rangle\langle\alpha|$ where $|\alpha\rangle$ ($\alpha \in \mathbf{C}$) is a coherent state defined by

$$|\alpha\rangle = e^{\alpha a^\dagger - \bar{\alpha}a} |0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle \quad (\Longleftarrow a|\alpha\rangle = \alpha|\alpha\rangle)$$

then

$$\tau(t) = (1 - G(t)) e^{|\alpha|^2 e^{-(\mu-\nu)t} \log G(t)} e^{-\log G(t)} \left\{ \alpha e^{-\left(\frac{\mu-\nu}{2} + i\omega_0\right)t} a^\dagger + \bar{\alpha} e^{-\left(\frac{\mu-\nu}{2} - i\omega_0\right)t} a - N \right\}. \tag{23}$$

The main part (which corresponds to the classical one)

$$\alpha e^{-\left(\frac{\mu-\nu}{2} + i\omega_0\right)t} a^\dagger + \bar{\alpha} e^{-\left(\frac{\mu-\nu}{2} - i\omega_0\right)t} a$$

appears in the formula, see Appendix. This derivation is not easy, so see [9] for further details.

[II] Second, we must calculate e^{tY} . We decompose Y into two parts

$$Y = -i\Omega(\tilde{Y}_1 - \tilde{Y}_2) \quad (24)$$

where

$$\begin{aligned} \tilde{Y}_1 &= \begin{pmatrix} 0 & 0 & a \otimes \mathbf{1} & 0 \\ 0 & 0 & 0 & a \otimes \mathbf{1} \\ a^\dagger \otimes \mathbf{1} & 0 & 0 & 0 \\ 0 & a^\dagger \otimes \mathbf{1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} & a \\ a^\dagger & \end{pmatrix} \otimes \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}, \\ \tilde{Y}_2 &= \begin{pmatrix} 0 & \mathbf{1} \otimes (a^\dagger)^T & 0 & 0 \\ \mathbf{1} \otimes a^T & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \otimes (a^\dagger)^T \\ 0 & 0 & \mathbf{1} \otimes a^T & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} \otimes \begin{pmatrix} & a \\ a^\dagger & \end{pmatrix}^T. \end{aligned} \quad (25)$$

From (24), (25) and [1] it is easy to see

$$\begin{aligned} e^{tY} &= \exp \left(-i\Omega t \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} \right) \otimes \exp \left(i\Omega t \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} \right)^T \\ &= \begin{pmatrix} \cos(\Omega t \sqrt{aa^\dagger}) & -i \frac{1}{\sqrt{aa^\dagger}} \sin(\Omega t \sqrt{aa^\dagger}) a \\ -i \frac{1}{\sqrt{a^\dagger a}} \sin(\Omega t \sqrt{a^\dagger a}) a^\dagger & \cos(\Omega t \sqrt{a^\dagger a}) \end{pmatrix} \otimes \\ &\quad \begin{pmatrix} \cos(\Omega t \sqrt{aa^\dagger}) & i \frac{1}{\sqrt{aa^\dagger}} \sin(\Omega t \sqrt{aa^\dagger}) a \\ i \frac{1}{\sqrt{a^\dagger a}} \sin(\Omega t \sqrt{a^\dagger a}) a^\dagger & \cos(\Omega t \sqrt{a^\dagger a}) \end{pmatrix}^T. \end{aligned} \quad (26)$$

[III] Third, we must calculate $e^{\frac{t^2}{2}[X,Y]}$. For the purpose we first calculate $[X, Y]$, which is relatively easy. Note that \tilde{Y}_1 and \tilde{Y}_2 commute from (25).

Then the result is

$$[X, Y] = -i\Omega \left\{ [X, \tilde{Y}_1] - [X, \tilde{Y}_2] \right\}$$

where

$$[X, \tilde{Y}_1] = \begin{pmatrix} 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \\ B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix}, \quad [X, \tilde{Y}_2] = \begin{pmatrix} 0 & C & 0 & 0 \\ D & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & D & 0 \end{pmatrix}$$

and

$$\begin{aligned} A &= \frac{\mu + \nu}{2} a \otimes \mathbf{1} - \nu \mathbf{1} \otimes a^T, & B &= -\frac{\mu + \nu}{2} a^\dagger \otimes \mathbf{1} + \mu \mathbf{1} \otimes (a^\dagger)^T, \\ C &= -\nu a^\dagger \otimes \mathbf{1} + \frac{\mu + \nu}{2} \mathbf{1} \otimes (a^\dagger)^T, & D &= \mu a \otimes \mathbf{1} - \frac{\mu + \nu}{2} \mathbf{1} \otimes a^T. \end{aligned}$$

It is easy to see that

$$[A, C] = [A, D] = 0, \quad [B, C] = [B, D] = 0,$$

so we can conclude that $[X, \tilde{Y}_1]$ and $[X, \tilde{Y}_2]$ commute.

Since

$$e^{\frac{t^2}{2}[X, Y]} = e^{-\frac{it^2}{2}\Omega[X, \tilde{Y}_1]} e^{\frac{it^2}{2}\Omega[X, \tilde{Y}_2]}$$

we can calculate each term easily. The result is

$$e^{-\frac{it^2}{2}\Omega[X, \tilde{Y}_1]} = \begin{pmatrix} \cos(\frac{t^2}{2}\Omega\sqrt{AB}) & 0 & -\frac{i}{\sqrt{AB}}\sin(\frac{t^2}{2}\Omega\sqrt{AB})A & 0 \\ 0 & \cos(\frac{t^2}{2}\Omega\sqrt{AB}) & 0 & -\frac{i}{\sqrt{AB}}\sin(\frac{t^2}{2}\Omega\sqrt{AB})A \\ -\frac{i}{\sqrt{BA}}\sin(\frac{t^2}{2}\Omega\sqrt{BA})B & 0 & \cos(\frac{t^2}{2}\Omega\sqrt{BA}) & 0 \\ 0 & -\frac{i}{\sqrt{BA}}\sin(\frac{t^2}{2}\Omega\sqrt{BA})B & 0 & \cos(\frac{t^2}{2}\Omega\sqrt{BA}) \end{pmatrix}$$

and

$$e^{\frac{it^2}{2}\Omega[X, \tilde{Y}_2]} = \begin{pmatrix} \cos(\frac{t^2}{2}\Omega\sqrt{CD}) & \frac{i}{\sqrt{CD}}\sin(\frac{t^2}{2}\Omega\sqrt{CD})C & 0 & 0 \\ \frac{i}{\sqrt{DC}}\sin(\frac{t^2}{2}\Omega\sqrt{DC})D & \cos(\frac{t^2}{2}\Omega\sqrt{DC}) & 0 & 0 \\ 0 & 0 & \cos(\frac{t^2}{2}\Omega\sqrt{CD}) & \frac{i}{\sqrt{CD}}\sin(\frac{t^2}{2}\Omega\sqrt{CD})C \\ 0 & 0 & \frac{i}{\sqrt{DC}}\sin(\frac{t^2}{2}\Omega\sqrt{DC})D & \cos(\frac{t^2}{2}\Omega\sqrt{DC}) \end{pmatrix}.$$

In last, we shall restore the result to original form. For the purpose, ignoring the term $e^{\frac{t^2}{2}[X,Y]}$ we set

$$\hat{\rho}(t) = e^{tY} e^{tX} \hat{\rho}(0) = e^{tY} \hat{\rho}_1(t), \quad \hat{\rho}_1(t) = e^{tX} \hat{\rho}(0) \quad (27)$$

and

$$\rho(0) = \begin{pmatrix} \rho_{00}(0) & \rho_{01}(0) \\ \rho_{10}(0) & \rho_{11}(0) \end{pmatrix}.$$

Then from [I] $\tilde{\rho}_1(t)$ becomes

$$\tilde{\rho}_1(t) = \begin{pmatrix} (11) & (12) \\ (21) & (22) \end{pmatrix} \quad (28)$$

where

$$\begin{aligned} (11) &= \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \{ \exp(\{-i\omega_0 t - \log(F(t))\}N) \times \\ &\quad \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho_{00}(\mathbf{0}) (a^\dagger)^m \right\} \exp(\{i\omega_0 t - \log(F(t))\}N) \} a^n, \\ (12) &= e^{-i\omega_0 t} \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \{ \exp(\{-i\omega_0 t - \log(F(t))\}N) \times \\ &\quad \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho_{01}(\mathbf{0}) (a^\dagger)^m \right\} \exp(\{i\omega_0 t - \log(F(t))\}N) \} a^n, \\ (21) &= e^{i\omega_0 t} \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \{ \exp(\{-i\omega_0 t - \log(F(t))\}N) \times \\ &\quad \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho_{10}(\mathbf{0}) (a^\dagger)^m \right\} \exp(\{i\omega_0 t - \log(F(t))\}N) \} a^n, \\ (22) &= \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \{ \exp(\{-i\omega_0 t - \log(F(t))\}N) \times \\ &\quad \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho_{11}(\mathbf{0}) (a^\dagger)^m \right\} \exp(\{i\omega_0 t - \log(F(t))\}N) \} a^n \end{aligned}$$

and from [II] $\tilde{\rho}(t)$ becomes

$$\begin{aligned} \tilde{\rho}(t) &= \begin{pmatrix} \cos(\Omega t \sqrt{N+1}) & -i \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}) a \\ -i \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) a^\dagger & \cos(\Omega t \sqrt{N}) \end{pmatrix} \tilde{\rho}_1(t) \times \\ &\quad \begin{pmatrix} \cos(\Omega t \sqrt{N+1}) & i \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}) a \\ i \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) a^\dagger & \cos(\Omega t \sqrt{N}) \end{pmatrix} \end{aligned}$$

or by making a slight modification in terms of $af(N) = f(N+1)a$

$$\begin{aligned} \tilde{\rho}(t) = & \begin{pmatrix} \cos(\Omega t \sqrt{N+1}) & -i \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}) a \\ -i \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) a^\dagger & \cos(\Omega t \sqrt{N}) \end{pmatrix} \tilde{\rho}_1(t) \times \\ & \begin{pmatrix} \cos(\Omega t \sqrt{N+1}) & ia \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) \\ ia^\dagger \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}) & \cos(\Omega t \sqrt{N}) \end{pmatrix}. \end{aligned} \quad (29)$$

By making use of this formula let us calculate an important example. The initial state is

Example

$$\rho(0) = \frac{1}{2} \begin{pmatrix} |0\rangle\langle 0| & \\ & |\alpha\rangle\langle \alpha| \end{pmatrix} \quad (30)$$

where $|\alpha\rangle$ is a coherent state in [I].

Then the result is

$$\tilde{\rho}_1(t) = \frac{1}{2} \begin{pmatrix} A & \\ & B \end{pmatrix} \quad (31)$$

where

$$\begin{aligned} A &= \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} e^{\log G(t)N}, \\ B &= (1 - G(t)) e^{|\alpha|^2 e^{-(\mu-\nu)t} \log G(t)} e^{-\log G(t) \left\{ \alpha e^{-\left(\frac{\mu-\nu}{2} + i\omega_0\right)t} a^\dagger + \bar{\alpha} e^{-\left(\frac{\mu-\nu}{2} - i\omega_0\right)t} a - N \right\}} \end{aligned}$$

(see (22) and (23)) and

$$\tilde{\rho}(t) = \frac{1}{2} \begin{pmatrix} (11) & (12) \\ (21) & (22) \end{pmatrix} \quad (32)$$

where

$$\begin{aligned} (11) &= \cos(\Omega t \sqrt{N+1}) A \cos(\Omega t \sqrt{N+1}) + \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}) a B a^\dagger \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}), \\ (12) &= i \cos(\Omega t \sqrt{N+1}) A a \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) - i \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}) a B \cos(\Omega t \sqrt{N}), \\ (21) &= -i \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) a^\dagger A \cos(\Omega t \sqrt{N+1}) + i \cos(\Omega t \sqrt{N}) B a^\dagger \frac{1}{\sqrt{N+1}} \sin(\Omega t \sqrt{N+1}), \\ (22) &= \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) a^\dagger A a \frac{1}{\sqrt{N}} \sin(\Omega t \sqrt{N}) + \cos(\Omega t \sqrt{N}) B \cos(\Omega t \sqrt{N}). \end{aligned}$$

These forms are compact and comparatively beautiful. Though we can of course calculate another example we stop here.

In this paper we reconsidered the Jaynes–Cummings model with dissipation from a different point of view and constructed a compact approximate solution when some initial condition was given. It is very fresh as far as we know. We will leave a further construction to readers who are interested in this topic. As for the preceding works see [12], [13] and [14], [15].

We conclude this paper by stating some future prospects. Our real target is the following master equation :

$$\frac{\partial}{\partial t}\rho = -i[H_R, \rho] + \mu \left\{ a\rho a^\dagger - \frac{1}{2}(a^\dagger a\rho + \rho a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho a - \frac{1}{2}(aa^\dagger \rho + \rho aa^\dagger) \right\}$$

where H_R is the Rabi Hamiltonian (without RWA (Rotating Wave Approximation)) given by

$$\begin{aligned} H_R &= \frac{\omega_0}{2}\sigma_3 \otimes \mathbf{1} + \omega_0 1_2 \otimes a^\dagger a + \Omega \sigma_1 \otimes (a + a^\dagger) \\ &= \begin{pmatrix} \frac{\omega_0}{2} + \omega_0 N & \Omega(a + a^\dagger) \\ \Omega(a + a^\dagger) & -\frac{\omega_0}{2} + \omega_0 N \end{pmatrix}. \end{aligned}$$

We call this **the Rabi model with dissipation**. The Jaynes–Cummings model (which is an approximate model with RWA) has some weak points (see for example [16] and its references), so we must treat a more realistic model like this. In the following paper(s) we will attack this model.

Appendix

In this appendix we review the solution of classical damped harmonic oscillator, which is important to understand the text. See any textbook on Mathematical Physics.

The differential equation is given by

$$\ddot{x} + \gamma\dot{x} + \omega^2x = 0 \quad (\gamma > 0) \quad (33)$$

where $x = x(t)$, $\dot{x} = dx/dt$ and the mass is set to 1 for simplicity. In the following we treat only the case $\omega > \gamma/2$ (the case $\omega = \gamma/2$ may be interesting).

Solutions (with complex form) are well-known to be

$$x_{\pm}(t) = e^{-\left(\frac{\gamma}{2} \pm i\sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2}\right)t},$$

so the general solution is given by

$$\begin{aligned} x(t) &= \left\{ \alpha e^{-\left(\frac{\gamma}{2} + i\sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2}\right)t} + \bar{\alpha} e^{-\left(\frac{\gamma}{2} - i\sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2}\right)t} \right\} x(0) \\ &= \left\{ \alpha e^{-\left(\frac{\gamma}{2} + i\omega\sqrt{1 - \left(\frac{\gamma}{2\omega}\right)^2}\right)t} + \bar{\alpha} e^{-\left(\frac{\gamma}{2} - i\omega\sqrt{1 - \left(\frac{\gamma}{2\omega}\right)^2}\right)t} \right\} x(0) \end{aligned} \quad (34)$$

where α is any complex number.

If $\gamma/2\omega$ is small enough we have an approximate solution

$$x(t) \approx \left\{ \alpha e^{-\left(\frac{\gamma}{2} + i\omega\right)t} + \bar{\alpha} e^{-\left(\frac{\gamma}{2} - i\omega\right)t} \right\} x(0). \quad (35)$$

References

- [1] K. Fujii and T. Suzuki : An Approximate Solution of the Jaynes–Cummings Model with Dissipation, to appear in Int. J. Geom. Methods Mod. Phys, **8**, No. 8, arXiv : 1103.0329 [math-ph].
- [2] K. Fujii, K. Higashida, R. Kato and Y. Wada : Cavity QED and Quantum Computation in the Weak Coupling Regime, J. Opt. B : Quantum and Semiclass. Opt, **6** (2004), 502, quant-ph/0407014.
- [3] K. Fujii, K. Higashida, R. Kato and Y. Wada : Cavity QED and Quantum Computation in the Weak Coupling Regime II : Complete Construction of the Controlled–Controlled NOT Gate, Trends in Quantum Computing Research, Susan Shannon (Ed.), **Chapter 8**, Nova Science Publishers, 2006 and Computer Science and Quantum Computing, James E. Stones (Ed.), **Chapter 1**, Nova Science Publishers, 2007, quant-ph/0501046.

- [4] H. -P. Breuer and F. Petruccione : The theory of open quantum systems, Oxford University Press, New York, 2002.
- [5] W. P. Schleich : Quantum Optics in Phase Space, WILEY-VCH, Berlin, 2001.
- [6] M. Scala, B. Militello, A. Messina, S. Maniscalco, J. Piilo and K.-A. Suominen : Cavity losses for the dissipative Jaynes-Cummings Hamiltonian beyond Rotating Wave Approximation, J. Phys. A: Math. Theor. **40** (2007), 14527, arXiv : 0709.1614 [quant-ph].
- [7] E. T. Jaynes and F. W. Cummings : Comparison of Quantum and Semiclassical Radiation Theories with Applications to the Beam Maser, Proc. IEEE, **51** (1963), 89.
- [8] R. Endo, K. Fujii and T. Suzuki : General Solution of the Quantum Damped Harmonic Oscillator, Int. J. Geom. Methods Mod. Phys, **5** (2008), 653, arXiv : 0710.2724 [quant-ph].
- [9] K. Fujii and T. Suzuki : General Solution of the Quantum Damped Harmonic Oscillator II : Some Examples, Int. J. Geom. Methods Mod. Phys, **6** (2009), 225, arXiv : 0806.2169 [quant-ph].
- [10] K. Fujii : Introduction to Coherent States and Quantum Information Theory, quant-ph/0112090.
- [11] C. Zachos : Crib Notes on Campbell-Baker-Hausdorff expansions, unpublished, 1999, see <http://www.hep.anl.gov/czachos/index.html>.
- [12] A. B. Klimov, S. M. Chumakov, J. C. Retamal and C. Saavedra : An algebraic approach to the Jaynes-Cummings model with dissipation, Phys. Lett. A, **211** (1996), 143.
- [13] C. Saavedra, A. B. Klimov, S. M. Chumakov and J. C. Retamal : Dissipation in collective interactions, Phys. Rev. A, **58** (1998), 4078.
- [14] K. Fujii : Algebraic Structure of a Master Equation with Generalized Lindblad Form, Int. J. Geom. Methods Mod. Phys, **5** (2008), 1033, arXiv : 0802.3252 [quant-ph].

- [15] K. Fujii : A Master Equation with Generalized Lindblad Form and a Unitary Transformation by the Squeezing Operator, arXiv : 0803.3105 [quant-ph].
- [16] Jonas Larson : On vacuum induced Berry phases, arXiv:1107.3447 [quant-ph].